Journal of Statistical Physics, Vol. 16, No. 2, 1977

Illustration of Reversed Causality with Remarks on Experiment

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Received June 9, 1976

The macroscopic states of two model systems are specified at two times and their approach to equilibrium for intervening times is studied. The models are the Kac ring and a certain automorphism on the torus. If the relaxation time is short compared to the interval for the boundary value problem, the systems are seen by explicit calculation to decay away from the initial state almost as if the final conditions had not been specified. As the systems approach the final time they exhibit normal decay in the reversed time variable. For longer relaxation times acausal effects may be observed. Some remarks are also included on experimental searches for the acausal effects of future boundary conditions.

KEY WORDS: Causality; time asymmetry.

1. INTRODUCTION

In this paper we examine some models whose solution may be of relevance to the thesis which connects thermodynamic and cosmological arrows of time. We shall also discuss observational and experimental tests which could presage a future contraction of the universe.

In setting up a mathematical system for study of time asymmetry some care must be exercised in the selection of boundary conditions. If one studies an initial value problem for mechanical systems, the thermodynamic arrow, if it points at all, points away from the initial time. To have any hope of demonstrating an arrow, one cannot prejudice the case at the outset by giving initial conditions, and instead should give boundary values at two separate times and study the behavior in between. This point of view has been put forth by a number of authors[1–4] and in some naturally time-sym-

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metric theories it in fact turns out to be difficult or impossible[2,5] to solve problems with the usual initial data.

In this paper we consider models in which macroscopic data are given at two times, say $t = 0$ and $t = T$, and for intervening times we examine the following questions:

1. Does the system go to equilibrium after time zero?
2. Does the system come out of equilibrium close to time T (reversed causality)?
3. Is the passage to equilibrium at early times affected by the future boundary conditions?
4. How are the answers to the foregoing questions affected if the relaxation time of the system is on the order of T?
5. What can these models say by analogy for physical systems, for example, long-lived nuclei?

The model to which we shall devote most of our attention is the Kac ring model.[6,7] In two appendices we examine boundary value problems for a decaying system in quantum mechanics and for a certain automorphism of the torus which has been found useful in studies of relaxation.

The possibility of finding experimental evidence for an imminent (5×10^{10} yr) collapse of the universe is discussed in a concluding section. In particular, one can use the Kac ring model to disprove the assertion that there would be a change in the laboratory-measured lifetime of long-lived nuclear species. Observational evidence for collapse turns out to be more subtle and we shall discuss some possibilities.

2. THE KAC RING MODEL

N balls are located at the N sites of a ring and each second all the balls move one step in the counterclockwise direction to occupy new sites. The balls are either black or white. Among the N sites there are M special sites which affect the dynamics of the balls' motion in the following way: When a ball *leaves* a special site it changes color. The set of special sites is called S.

The following notation (mostly that of Kac) is useful: $p, p = 1,..., N$, labels sites; equations in p are mod N;

$$\epsilon_p = \begin{cases} -1 & \text{if } p \in S \\ +1 & \text{if } p \notin S \end{cases}$$

$$\eta_p(t) = \begin{cases} +1 & \text{if the ball at site } p \text{ at time } t \text{ is white} \\ -1 & \text{if the ball at site } p \text{ at time } t \text{ is black} \end{cases}$$

$N_W(t)$ is the total number of white balls at time t; $N_B(t)$ is the total number of black balls at time t;

$$D(t) = (1/N)[N_W(t) - N_B(t)] = (1/N) \sum_p \eta_p(t)$$

is the fractional excess of white balls at time t. $\mu = M/N$ is the density of S, assumed to satisfy $0 < \mu < \frac{1}{2}$. In terms of ϵ and η the dynamics of the system are given by

$$\eta_p(t) = \epsilon_{p-1}\eta_{p-1}(t), \qquad p = 1,\dots, N \tag{1}$$

Iterating the foregoing gives

$$\eta_p(t) = \epsilon_{p-1}\epsilon_{p-2}\cdots\epsilon_{p-t}\eta_{p-t}(0) \tag{2}$$

The exact expression for $D(t)$ is

$$D(t) = \frac{1}{N}\sum_{p=1}^{N} \epsilon_{p+t-1}\cdots\epsilon_p\eta_p(0) \tag{3}$$

We shall not reproduce the molecular chaos argument (*Stosszahlansatz*) showing that $D(t) \to 0$ and the associated discussion of the recurrence and reversibility paradoxes. These can be found in Kac's book. Instead we shall immediately perform the ensemble averaging of S needed to obtain relaxation of the system.

Consider an ensemble of dynamical systems with different possible sets S. For each member of the ensemble the set, S is specified through flipping a biased coin N times, once for each site. In effect, ϵ_p is a random variable with

$$\text{prob}(\epsilon_p = -1) = \mu, \qquad \text{prob}(\epsilon_p = +1) = 1 - \mu \tag{4}$$

$$\epsilon_p, \epsilon_{p'} \text{ are independent for } p \neq p'$$

Clearly

$$\langle \epsilon_p \rangle = 1 - 2\mu \equiv e^{-\gamma} \qquad \text{all } p \tag{5}$$

which defines γ.

The ensemble average of $D(t)$ can now be computed (for $t < N$)

$$\begin{aligned}\langle D(t)\rangle &= \frac{1}{N}\sum_{p=1}^{N} \langle \epsilon_{p+t-1}\cdots\epsilon_p\rangle\eta_p(0) \\ &= \frac{1}{N}\sum_{p=1}^{N} \langle \epsilon_{p+t-1}\rangle\cdots\langle\epsilon_p\rangle\eta_p(0) \\ &= (1 - 2\mu)^t \sum_{p=1}^{N} \eta_p(0) = e^{-\gamma t}D(0)\end{aligned} \tag{6}$$

which gives exponential decay of D. The restriction to $t < N$ occurs since otherwise squares of the ϵ's appear in the sum. This is a reflection of the fact that even after ensemble averaging there is still a Poincaré recurrence with recurrence time $2N$. The linear dependence of the recurrence time on the number of "particles," rather than a more physical exponential dependence, occurs because the dynamics of the system, for given initial conditions, restricts the system to a small subset of the 2^N possible states.

3. TWO-TIME BOUNDARY CONDITIONS

We consider systems for which D assumes given values at two times

$$D(0) = \alpha \tag{7}$$

$$D(T) = \beta \tag{8}$$

While T is to be considered a large number (analog of age of the universe), it is still small compared to $2N$, the Poincaré recurrence time. Clearly once the state of the system is given at $t = 0$ its value at $t = T$ is fixed and therefore only a small fraction of those initial conditions for which (7) is true will also satisfy (8). Our method is to do an ensemble average not only over the set S, but also over those initial sequences $\{\eta_p(0)\}$ that satisfy *both* (7) and (8) on the average.

For this double averaging to make sense it must be done sequentially, first over $\{\eta_p(0)\}$ for fixed S and then over S. Suppose S is fixed, and consider "microscopic" data $\eta_p(0)$ satisfying

$$\alpha = (1/N) \sum_{p=1}^{N} \eta_p(0) \tag{9}$$

$$\beta = (1/N) \sum_{p=1}^{N} \epsilon_{p+T-1} \cdots \epsilon_p \eta_p(0) \tag{10}$$

For convenience define

$$\phi_p = \eta_p(0), \qquad \delta_p = \epsilon_{p+T-1} \cdots \epsilon_p \tag{11}$$

As in the selection of S, we shall look not only at sequences that satisfy (9) and (10) exactly, but at others also; ϕ_p is selected by another random process designed to guarantee only that

$$\alpha = (1/N) \sum_p \langle \phi_p \rangle \tag{12}$$

$$\beta = (1/N) \sum_p \delta_p \langle \phi_p \rangle \tag{13}$$

To motivate the process used to fix ϕ_p we consider some definite sequence ϕ_p, $p = 1,\ldots, N$, satisfying (9) and (10); these constraints separately determine the number of $(+1)$'s and (-1)'s among the ϕ_p for which $\delta_p = +1$ and among those for which $\delta_p = -1$. Let

$$C = \{p \mid \delta_p = +1\} \tag{14}$$

The set C is fixed by S. The symbol C also stands for the number of elements in the set C. Let

$$x = \frac{1}{C}\sum_{p\in C}\phi_p, \qquad y = \frac{1}{N-C}\sum_{p\notin C}\phi_p \tag{15}$$

Then by (9) and (10)

$$\alpha = cx + c'y, \qquad \beta = cx - c'y \tag{16}$$

where $c = C/N$ and $c' = 1 - c$. It follows that if a random process is to be used in selecting ϕ_p, Eq. (15) should be satisfied on the average, and therefore the probabilities used for picking ϕ_p, $p \in C$, should be different (since in general $x \neq y$) from those used for $p \notin C$. Let ϕ_p for $p \in C$ have probability ξ of taking the value -1, ϕ_p for $p \notin C$ have probability η of taking the value -1, and let ϕ_p and $\phi_{p'}$ be independent for $p \neq p'$. From (15) it follows that

$$x = 1 - 2\xi, \qquad y = 1 - 2\eta \tag{17}$$

This assignment of probabilities guarantees that the average behavior of sequences in the macroscopically defined ensemble is the same as that of the sequences in an ensemble defined by requiring the conditions (7) and (8) exactly.

Not all boundary conditions admit of solution. The inequalities

$$0 \leqslant \xi \leqslant 1, \qquad 0 \leqslant \eta \leqslant 1 \tag{18}$$

restrict α and β to the rectangle

$$|\alpha + \beta| \leqslant 2c, \qquad |\alpha - \beta| \leqslant 2c' \tag{19}$$

The object of interest is

$$\bar{D}(t) \equiv \langle\langle D(t)\rangle_\phi\rangle_\epsilon = \left\langle\left\langle\frac{1}{N}\sum_p \phi_p\epsilon_p\cdots\epsilon_{p+t-1}\right\rangle_\phi\right\rangle_\epsilon \tag{20}$$

where the appropriate averaging process is indicated by a subscript ϕ or ϵ. The first and essentially only probabilistically interesting aspect of our calculation has now made its appearance, in that the variable ϕ_p is not independent of the ϵ's, because the selection of ϕ_p depended on $\delta_p = \epsilon_p\cdots\epsilon_{p+T-1}$. This dependence is exactly what makes our calculation work.

The sum over p is broken into that over C and that over its complement, and averaging of ϕ is performed. Then

$$\bar{D}(t) = x\left\langle\frac{1}{N}\sum_{p\in C}\epsilon_p\cdots\epsilon_{p+t-1}\right\rangle + y\left\langle\frac{1}{N}\sum_{p\notin C}\epsilon_p\cdots\epsilon_{p+t-1}\right\rangle \tag{21}$$

The remaining brackets in (21) are conditional expectation values and so as to simplify their calculation we make the following definitions:

$$E = \{p \mid \epsilon_p \epsilon_{p+1} \cdots \epsilon_{p+t-1} = 1\}, \qquad F = \{p \mid \epsilon_{p+t} \cdots \epsilon_{p+T-1} = 1\} \tag{22}$$

By the definition of δ_p

$$C = (E \cap F) \cup (E' \cap F') \tag{23}$$

(prime is complementation). To evaluate

$$\langle \epsilon_p \cdots \epsilon_{p+t-1} \rangle_{\delta_p = +1} \tag{24}$$

[which is what appears in the first sum in (21)] we need the measures of the sets[2] $E \cap C$ and $E \cap F$. In terms of these

$$\langle \epsilon_p \cdots \epsilon_{p+t-1} \rangle_{\delta_p = +1} = \frac{P(E \cap C)(+1) + P(E' \cap C)(-1)}{P(C)} \tag{25}$$

By (23) $E \cap C = E \cap F$ and $E' \cap C = E' \cap F'$, and since E and F are independent, it follows that

$$\langle \epsilon_p \cdots \epsilon_{p+t-1} \rangle_{\delta_p = +1} = \frac{P(E)P(F) - [1 - P(E)][1 - P(F)]}{P(C)} \tag{26}$$

Similarly, since $C' = (E \cap F') \cup (E' \cap F)$, we have

$$\langle \epsilon_p \cdots \epsilon_{p+t-1} \rangle_{\delta_p = -1} = \frac{P(E)[1 - P(F)] - [1 - P(E)]P(F)}{P(C')} \tag{27}$$

The probabilities $P(E)$, $P(F)$, and $P(C)$ are not conditional and can simply be evaluated from (taking E as an example)

$$\begin{aligned} \langle \epsilon_p \cdots \epsilon_{p+t-1} \rangle &= \langle \epsilon_p \rangle \cdots \langle \epsilon_{p+t-1} \rangle = e^{-\gamma t} \\ &= P(E)(+1) + [1 - P(E)](-1) \end{aligned} \tag{28}$$

so that

$$P(E) = \tfrac{1}{2} + \tfrac{1}{2}e^{-\gamma t}, \qquad P(F) = \tfrac{1}{2} + \tfrac{1}{2}e^{-\gamma(T-t)}, \qquad P(C) = \tfrac{1}{2} + \tfrac{1}{2}e^{-\gamma T} = c \tag{29}$$

These results are substituted in (21) and after some algebra one obtains

$$\begin{aligned} \bar{D}(t) &= \frac{\alpha - \beta e^{-\gamma T}}{1 - e^{-2\gamma T}} e^{-\gamma t} + \frac{\beta - \alpha e^{-\gamma T}}{1 - e^{-2\gamma T}} e^{-\gamma(T-t)} \\ &= (\sinh \gamma T)^{-1}\{\alpha \sinh[\gamma(T - t)] + \beta \sinh(\gamma t)\} \end{aligned} \tag{30}$$

[2] Strictly speaking, we are not looking at the sets $E \cap C$, etc., themselves, but rather at measures of sets in an implicitly understood sample space Ω (indexed by ω). Thus by $P(E \cap C)$ is actually meant "measure of the set in Ω for which $\phi_p(\omega)$ is such that $p \in E \cap C$."

Equation (30) is the main result of this paper. $\bar{D}(t)$ is the sum of two exponentials, one dropping off as t increases and one increasing as t approaches T. For $\gamma T \gg 1$, during the system's early evolution it decays in a normal fashion away from the initial value $\bar{D}(0) = \alpha$ so long as $e^{-\gamma T}$ is ignored. On the approach to T the term $e^{-\gamma(T-t)}$ dominates and, if one were to look at things in the time-reversed variable $\tau = T - t$, a normal decay away from $\tau = 0$ would be observed. To an observer who through some quirk of fate managed to have his time sense in the direction t but nevertheless was living in the era just before T, a strange scene would unfold. Although the microscopic behavior of each ball and site would be unimpeachable, there would seem to be some conspiracy afoot to form a systematically more ordered state as the function $D(t)$ increased its value toward $\beta = \bar{D}(T)$. (We are implicitly assuming the easily verified fact that $\Delta D = \{[D(t) - \bar{D}(t)]^2\}^{1/2}$ goes as $1/\sqrt{N}$.) It would look like a movie of breaking eggs run backward.

To continue the discussion for nonzero $e^{-\gamma T}$, we simplify a bit by taking $\alpha = \beta$, so that

$$\bar{D}(t) = \frac{\alpha \cosh[\gamma(\frac{1}{2}T - t)]}{\cosh(\gamma T/2)} \tag{31}$$

This is the hyperbolic cosine anticipated by Wheeler.[8] The minimum value of $\bar{D}$ is $[\cosh(\gamma T/2)]^{-1}$. If $e^{-\gamma T}$ is not negligible, the curve is seen to depart effectively from its exponential decay. In particular, one might define an effective decay rate from the ratio

$$\Gamma = -(1/\bar{D})\, \partial\bar{D}/\partial t \tag{32}$$

From (31)

$$\Gamma = \gamma \tanh[\gamma(\tfrac{1}{2}T - t)] \tag{33}$$

If for most times the argument of tanh is large, the system decays or "reverse decays" most of the time. If γT is not large, the decay rate for $\bar{D}$ never reaches the value it has for the initial value problem.

4. REMARKS ON EXPERIMENT

Before saying anything positive, we would like to use the Kac ring model to eliminate one kind of experiment that one might have thought could be used to estimate the time T.

For purposes of analogy, the time T should be thought of as the period for a single cosmological oscillation. Just to fix numbers, we shall take this as being on the order of 60×10^9 years, in line with Wheeler's estimate.[8] To see departures from exponential decay one needs phenomena with relaxation times on this order and certain unstable nuclear species do have the required lifetimes. One candidate might be Sm^{147}, which has a half-life of about 10^{11} yr and is also reasonably abundant. Another possibility is Rb^{87}. So as not to prejudice the discussion, we shall refer to the unstable species as A and let its decay be $A \to B + C$. We assume that dynamically A should decay according to $e^{-\gamma t}$.

Despite the suggestiveness of Eq. (33), if one measures the lifetime of a sample of A in the laboratory, he will see the decay rate γ and *not* $\gamma \tanh[\gamma(\frac{1}{2}T - t)]$. The future collapse of the universe will not influence the laboratory results, not now ($t < T/2$ presumably) and not even for $t > T/2$ if one could somehow survive past $T/2$ with time sense intact. (Throughout this paper we assume that consciousness and the thermodynamic arrow ought to go in the same direction. Although we phrase our discussion in terms of this reasonable though difficult to prove hypothesis, our results in no way depend on it.)

The laboratory experiment on the sample of A is modeled in Kac's ring as follows: Suppose that at some time T_1 the sites $42, 43, \ldots, 42 + K - 1$ are all occupied by white balls. We want to predict the subsequent behavior of those K balls. The number K (lab sample $\sim 10^{24}$) is assumed very small compared to N (total number of A's in the universe, say 10^{65}). To study this question in our earlier model we must build the additional condition into our ensemble. Thus the sequences ϕ_p must satisfy, besides Eqs. (12) and (13), the additional condition that

$$\eta_p(T_1) = \epsilon_{p-1} \cdots \epsilon_{p-T_1+1} \phi_{p-T_1+1} = 1 \qquad \text{for } p = 42, \ldots, 42 + K - 1 \tag{34}$$

But now the answer to our question is trivial. For given S, the condition (34) fixes K of the ϕ_p's (specifically ϕ_q with $q = 41 - T_1, \ldots, 42 - T_1 + K \bmod N$) precisely and no coin is to be flipped for these K balls. Withdrawing these K balls from the random process has no discernible effect on the selection of the remaining $N - K$ balls if K is small enough compared to N. Concerning the K balls of interest, we are given that they are white at $t = T_1$ and when the averaging over S is performed they will decay exactly as in the usual initial value problem for the Kac ring model.

The function $\bar{D}(t)$ therefore does not say much about the behavior of small, specially selected samples, but rather is directed to the overall abundance of A in the universe. An estimate of T must then come from estimates of the overall abundance of A as a function time. Unfortunately, here, too, there are problems. According to the present picture of heavy element production, creation of A is going on all the time in stars by processes $Z \to A$ other than $A \rightleftarrows B + C$. Thus the boundary value problem with large initial and final values is not a good description of A. What is needed is a species that early in the universe's history comes into existence by a process that ceases to operate subsequently (and until the universe is dense enough near the end of its cycle for the reverse process). When creation of A by this process is finished, A $\rightleftarrows$ B + C takes over. There may be no long-lived nucleus that fits this picture. Furthermore, even granted such a species, measurement of the time dependence of overall abundance with the accuracy needed to distinguish cosh from exp does not seem practicable.

There are other slowly relaxing processes in the universe. The dynamics of a cluster of galaxies may be a place to look for a system which is not coming to equilibrium as fast as the usual statistical mechanics says it ought. The problems here are that (1) the mechanics of clusters are not fully understood, (2) the relaxation times may not be long enough, and (3) in principle a system in free space bound by its own attractive forces cannot be in perfect equilibrium because particles at the tail of the Maxwell–Boltzmann distribution can escape.

5. CONCLUSION

5.1. Experiments

The kind of experiment proposed above makes sense even if one rejects our model and the philosophy behind it. One can think simply in terms of checking on whether we have an oscillating cosmology and whether the state of the universe is roughly the same at the beginning and end of each oscillation. If the latter is indeed the case and if $f(t)$ is some cosmological dynamical quantity, then $f(t)$ should be symmetric about $T/2$ (T = period of oscillation). An experimental test then consists in finding a system for which symmetric behavior of f is inconsistent with the evolution of f as predicted by the usual statistical mechanics. Using Wheeler's[8] figures and our model, the best f would be one with a relaxation time of about 40×10^9 yr, for which the exp and cosh would differ by about 11%.

The experiments suggested earlier are probably not practicable. However, inasmuch as an estimate of T as contemplated in this article is logically distinct from measurements of T from deceleration of cosmological expansion, it seems that a more feasible experimental test would be of great interest.

5.2. Kac Ring Model

Our model calculation, with its hyperbolic cosine time dependence, will not come as a surprise to many of those who have thought about this problem. However, in view of the fact that the obvious is not always easily provable and may even be controversial we feel there is a real benefit in having a fully solved model. One immediate benefit of this model was in answering a question about laboratory measurements of lifetimes.

5.3. Entropy in Oscillating Universes

In passing, we wish to reconcile calculations showing entropy increase in oscillating universes, such as that reported in Ref. 9, with the symmetric view represented in this paper and in Refs. 4 and 10. If one assumes *ab initio* that there is never any switch in the thermodynamic arrow and calculates accordingly, bringing in light absorption and other irreversible processes where necessary, then it is understandable that increase in entropy within an oscillation and in successive oscillations can result. On the other hand, with symmetric boundary conditions the usual causality and with it entropy production are reversed toward the end of an oscillation and therefore explicit calculations based on irreversible equations cannot be continued into that era. (Entropy in the Kac ring model can easily be defined and behaves as expected.) Our conclusion is that logically speaking both symmetric and nonsymmetric approaches to the problem are consistent and it is experiment that must choose between them.

APPENDIX A. BOUNDARY VALUE PROBLEM FOR AUTOMORPHISMS ON A TORUS

Phase space for this system is the two-dimensional torus of radii one and the dynamics are given by the automorphism

$$x(t+1) = x(t) + y(t) \mod 1, \qquad y(t+1) = x(t) + 2y(t) \mod 1 \tag{A.1}$$

This system has been extensively studied for its mixing properties[11] and also in two-time boundary value problems.[12] The method we shall now present leads to simplification of the boundary value problem. Rewrite (A.1) as

$$\xi = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad \xi(t+1) = M\xi(t) \mod 1, \qquad M = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \tag{A.2}$$

Because M has integer entries, we can write simply

$$\xi(t) = M^t \xi(0) \mod 1 \tag{A.3}$$

with the mod 1 operation taken at the end.

Let there be a number of initial points ξ_i, $i = 1,..., N$. Let

$$\rho(\xi, t) = \frac{1}{N} \sum_{i=1}^{N} \delta(\xi - \xi_i(t)) \tag{A.4}$$

The two-dimensional delta function on the torus is

$$\delta(\xi) = \sum_{n,m=-\infty}^{\infty} \exp[2\pi i(xn + ym)] \equiv \sum_{\nu} \exp(2\pi i\bar{\nu}\xi)$$

$$\int d^2\xi\, \delta(\xi) = \int_0^1 dx \int_0^1 dy\, \delta(\xi) = 1 \tag{A.5}$$

where the bar on ν indicates transpose and $\nu = \begin{pmatrix} n \\ m \end{pmatrix}$.

The advantage of using (A.4) with the periodic delta function is that the mod 1 in (A.3) can be ignored and we have

$$\rho(\xi, t) = \frac{1}{N} \sum_i \sum_\nu \exp[2\pi i\bar{\nu}(\xi - M^t\xi_i)] \tag{A.6}$$

From this we identify the Fourier coefficients of ρ (the tilde denotes Fourier transform)

$$\tilde{\rho}_\nu(t) = \frac{1}{N} \sum_i \exp(-2\pi i\bar{\nu}M^t\xi_i) \tag{A.7}$$

The mixing property of the system is reflected in the rapid growth of the matrix elements of M^t. The time dependence of $\tilde{\rho}_\nu(t)$ is obviously (using $\bar{M} = M$)

$$\tilde{\rho}_\nu(t) = \tilde{\rho}_{M^t\nu}(0) \tag{A.8}$$

Consequently, if the initial distribution is smooth enough that $\tilde{\rho}_\nu(0) \sim 1/|\nu|^k$, then a given Fourier coefficient ν_0 will drop off according to the rule (A.8). The elements of M^t grow according to

$$(M^t)_{ij} \sim e^{\gamma t}, \qquad \gamma = \log[(3 + \sqrt{5})/2] \cong 0.96 \tag{A.9}$$

so that we can expect $\tilde{\rho}_\nu(t) \sim \exp(-\gamma kt)$. The only exception to this rule is $\nu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and we have

$$\rho(\xi, t) \to 1 \quad \text{as} \quad t \to \infty \tag{A.10}$$

A number of possibilities suggest themselves for the two-time boundary condition problem. For example, one could consider specifying certain moments of ρ initially and finally. This turns out to be trivial and gives a symmetric time dependence which does not use the mixing property of the system.

The boundary information that we shall give is that the system points were contained in a measurable set A initially and in another measurable set B finally. All configurations satisfying these conditions will be given equal weight. Let A_t represent the image of the set A under the tth power of the automorphism and for a set C let $\chi_C(\xi)$ be the characteristic function of the set. Let μ be Lebesgue measure. The mixing property states that

$$\lim_{t\to\infty} \mu(A_t \cap B) = \mu(A)\mu(B) \tag{A.11}$$

which in view of our various definitions is equivalent to

$$\lim_{t\to\infty} \sum_{\substack{\nu \\ \nu\neq 0}} \tilde{\chi}^*_{B\nu}\tilde{\chi}_{A_t\nu} = \lim_{t\to\infty} \sum_{\substack{\nu \\ \nu\neq 0}} \tilde{\chi}^*_{B\nu}\tilde{\chi}_{A,M^t\nu} = 0 \tag{A.12}$$

That is, except for the term $\nu = 0$, a sum of terms $\tilde{\chi}^*_{B\nu}\tilde{\chi}_{A,M^t\nu}$ goes to zero. This does depend on the details of M since it is the rapid growth of M and the irrationality of its eigenvectors that guarantee this. For fixed ν it is not hard (see Ref. 11, Section 10) to use (A.12) to get

$$\sum_{\substack{\sigma \\ \sigma\neq 0}} \tilde{\chi}_{B,\nu-\sigma}\tilde{\chi}_{A,M^t\sigma} \to 0 \quad \text{as} \quad t\to\infty \tag{A.13}$$

To satisfy the boundary conditions given above, imagine that points in A are examined to see whether they get to B at time T. All points that do are kept. Thus if at $t = 0$, ρ is proportional to χ_A, at $t = T$ it should also be proportional to χ_B. The density function satisfying this (and which is not properly normalized) is

$$\rho(\xi, 0) = \chi_A(\xi)\chi_{B_{-T}}(\xi) \tag{A.14}$$

The Fourier coefficients of ρ, with their time dependence, are

$$\tilde{\rho}_\nu(t) = \tilde{\rho}_{M^t\nu}(0) = \sum_\sigma \tilde{\chi}_{A,M^t\nu-\sigma}\tilde{\chi}_{B,M^{-T}\sigma} \tag{A.15}$$

For fixed ν, such that $M^t\nu$ is not too large, and for large T, the only significant contribution to the sum will be from $\sigma = 0$. This is just the mixing property as expressed in Eq. (A.13). But then the behavior of ρ is

$$\tilde{\rho}_\nu(t) \sim \tilde{\chi}_{A,M^t\nu}\mu(B) \tag{A.16}$$

since $\mu(B) = \tilde{\chi}_{B0}$. Thus if $\tilde{\rho}_\nu$ is normalized by dividing by $\mu(B)$, the time dependence looks just like the usual decay away from the density χ_A. For later times t one can expect all terms in the sum (A.15) to be small until as t approaches T we expect ρ to become proportional to χ_B. This can be seen through rewriting (A.15) as (with a few changes of variables)

$$\tilde{\rho}_\nu(T - \tau) = \sum_\sigma \tilde{\chi}_{A,M^T\sigma}\tilde{\chi}_{B,M^{-\tau}\nu-\sigma}$$

For large T and small τ the only significant term is again $\sigma = 0$.

The interesting situation is when T is short enough (or one is looking at high enough Fourier coefficients) to see noncausal effects, e.g., effects of future boundary conditions during the period $t < T/2$. The sum (A.15) is again rewritten as

$$\tilde{\rho}_\nu(t) = \sum_\sigma \tilde{\chi}_{A,M^t\nu - M^T\sigma}\tilde{\chi}_{B\sigma} \tag{A.17}$$

Seeing future effects corresponds to components of $\tilde{\chi}_B$ other than $\sigma = 0$ being significant. These will contribute most to the sum for those σ for which $M^T\sigma$ is as small as possible, so that $M^t\nu - M^T\sigma$ can be as small as possible. The matrix M has eigenvalues e^γ and $e^{-\gamma}$ [Eq. (A.9)] and $M^t\sigma$ will grow most slowly for σ close to the eigenvector of M with eigenvalue $e^{-\gamma}$. Of course the irrationality of the components of this vector prevents σ, with integer components, from being an exact eigenvector. From a practical point of view approximate eigenvectors do not buy too much time. Even for as good an approximation as

$$-\tfrac{1}{2}(\sqrt{5} + 1) - (-610/377) \sim 3 \times 10^{-6}$$

it only takes seven steps before the small component in the direction of the other eigenvector reaches appreciable size (~ 1).

The message to be gotten from (A.17) is that the most persistent (in minus t) effects of the future are the Fourier coefficients of modes σ with slow growth under M. In physical terms the conclusion is that one should look for phenomena with very slow relaxation times. This is an obvious conclusion and reaching it hardly requires all the foregoing formalism. However, for many of the topics treated in this paper it is worthwhile to provide justification for certain "obvious" points of view.

APPENDIX B. BOUNDARY VALUE PROBLEMS IN QUANTUM MECHANICS

In this case we do not have a neat explicit solution and our main interest is in formulating the boundary value problem. Having done so, we shall argue heuristically concerning solutions.

Since the dynamical equation of quantum mechanics is first order in time, there is no natural way to give two-time boundary conditions as in classical mechanics. We shall suggest a particular boundary value problem for which, as is often the case in these circumstances, there is no guarantee of existence or uniqueness of solution.

Given some basis $|\alpha\rangle$, $\alpha = 0, 1, \ldots, N$, the ensemble will be specified by the diagonal elements of the density matrix at two times:

$$\langle\alpha|\rho(0)|\alpha\rangle, \qquad \langle\alpha|\rho(T)|\alpha\rangle \tag{B.1}$$

To see how this can give reversed causality, suppose the system has the property that for the initial value problem, if the system is started in the state $|0\rangle$, it decays exponentially and spreads evenly among the N states. Call this solution of the wave equation $|\psi_R\rangle$. Models of such systems are discussed in Refs. 13 and 14. We assume as usual that $T \ll$ recurrence time. Consider the specific boundary values

$$\begin{aligned} \langle 0|\rho(t)|0\rangle &= \tfrac{1}{2} \\ \langle \alpha|\rho(t)|\alpha\rangle &= 1/2N, \qquad \alpha = 1,\dots, N \end{aligned} \tag{B.2}$$

for $t = 0$ and $t = T$. One state in (or almost in) this ensemble is

$$\psi = (\psi_R + \psi_A)/\sqrt{2} \tag{B.3}$$

where ψ_A is the reversed solution, given for real Hamiltonian and real basis functions by $\psi_A(t) = \psi_R{}^*(T - t)$. Even if ψ does not exactly satisfy (B.2), we shall assume that there are many wave functions near it (in norm) which do. It is true that some of these states may have large components in the direction $|0\rangle$ or in some other direction for times t, $0 < t < T$, but this ought to be about as likely as a Poincaré recurrence. Consequently, when an ensemble average is performed the overall behavior of the ensemble should be exponential decay away from $|0\rangle$ after $t = 0$ and growth back to $|0\rangle$ as t approaches T.

ACKNOWLEDGMENTS

I wish to thank A. Finzi, C. Kuper, A. Mann, C. Newman, R. Opher, M. Revzen, N. Rosen, B. Rosner, and R. Shtokhamer for helpful and stimulating discussions.

REFERENCES

1. L. S. Schulman, *Phys. Rev. D* **7**:2869 (1973).
2. L. S. Schulman, *J. Math. Phys.* **15**:295 (1974).
3. W. J. Cocke, *Phys. Rev.* **160**:1165 (1967).
4. H. Schmidt, *J. Math. Phys.* **7**:494 (1966).
5. L. S. Schulman, Formulation and Justification of the Wheeler–Feynman Absorber Theory, preprint.
6. M. Kac, *Probability and Related Topics in Physical Sciences*, Interscience (1959), Sections 14 and 15.
7. M. Coopersmith and G. Mandeville, *J. Stat. Phys.* **10**:391 (1974).
8. J. A. Wheeler, in *General Relativity and Gravitation* (GR7), G. Shaviv and J. Rosen, eds., Wiley, New York and Israel University Press, Jerusalem (1975).
9. P. T. Landsberg and D. Park, *Proc. Roy. Soc. Lond. A* **346**:485 (1975).
10. D. L. Schumacher, *Proc. Camb. Phil. Soc.* **60**:575 (1964).

11. V. I. Arnold and A. Avez, *Ergodic Problems of Classical Mechanics*, Benjamin (1968).
12. L. S. Schulman and R. Shtokhamer, Thermodynamic Arrow for a Mixing System, preprint.
13. G. C. Stey and R. W. Gibberd, *Physica* **60**:1 (1972).
14. P. C. W. Davies, *The Physics of Time Asymmetry*, Surrey Univ. Press, London (1974).